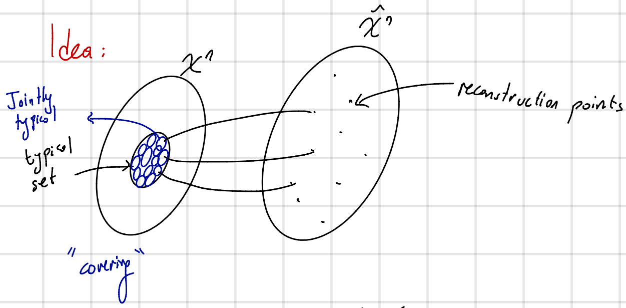
10/11/2016
Tuesday

Idea:



Size of the typical set: $2^{nH(X)}$

Size of ball $2^{nH(X|\hat{X})}$

\# of balls = $2^{nI(X;\hat{X})}$

Strong typical set: $T_\varepsilon^n = \left\{ x^n : |\pi(x|x^n) - P_X(x)| \le \varepsilon \frac{P_X(x)}{H(x)} \right\}$

Properties of Strong typical set:

1. If X^n is i.i.d. $\sim P_X$ then $\mathbb{P}[T_\varepsilon^{(n)}] \to 1$

2. $T_\varepsilon^{(n)} \in A_\varepsilon^{(n)}$

3. If $(x^n, y^n) \in T_\varepsilon^{(n)}(X,Y)$ then $x^n \in T_\varepsilon^{(n)}(X)$ and $y^n \in T_\varepsilon^{(n)}(Y)$

$$\Rightarrow T_\varepsilon^{(n)}(X,Y) \in A_\varepsilon^{(n)}(X,Y)$$

proof of 3. Given $\left|\pi(x,y|x^n,y^n) - P_{XY}(x,y)\right| \le \varepsilon \frac{P_{XY}(x,y)}{H(X,Y)}$

$$|\pi(x|x^n) - P_X(x)| = \left| \sum_y \pi(x,y|x^n,y^n) - \sum_y P_{XY}(x,y) \right| \le \sum_y \left|\pi(x,y|x^n,y^n) - P_{XY}(x,y)\right|$$

$$\le \sum_y \varepsilon \frac{P_{XY}(x,y)}{H(X,Y)} = \frac{\varepsilon P_X(x)}{H(X,Y)} = \frac{\varepsilon P_X(x)}{H(X)} \checkmark$$

4. If $x^n \in T^{(n)}$

$$\frac{1}{n}\sum_{i=1}^n g(x_i) \in \left[(1-\varepsilon)\,\mathbb{E}_{P_X} g(X),\ (1+\varepsilon)\,\mathbb{E}_{P_X} g(X)\right] \quad \text{(same proof as 2.)}$$

5. $|T_\varepsilon^{(n)}| \in \left[(1-\varepsilon)\,2^{-n(H(X)+\varepsilon)},\ 2^{-n(H(X)-\varepsilon)}\right]$

n large enough.

Combine property 1 and 2.

inherits (use 2.)

same as weak typical set counter part.

6. If (X^n, Y^n) i.i.d. $\sim P_X P_Y$, $\mathbb{P}[(X^n, Y^n) \in T_\varepsilon^{(n)}] \in \left[(1-\varepsilon)\,2^{-n(I(X;Y)+3\varepsilon)},\ 2^{-n(I(X;Y)-3\varepsilon)}\right]$

n large enough

use 1, 2 and 5.

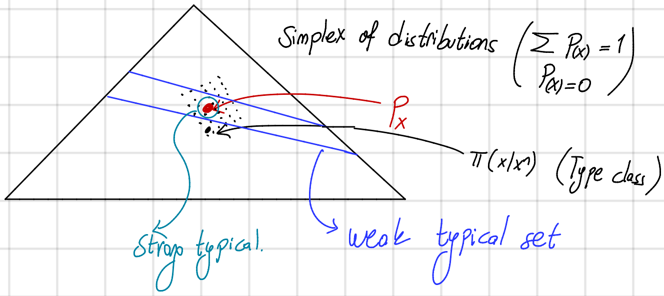
7. Conditionally typical set

$$T_\varepsilon^{(n)}(Y|x^n) = \left\{ y^n : (x^n, y^n) \in T_\varepsilon^{(n)} \right\}$$

7. a.) If $Y^n \sim \prod_{i=1}^{n} P_{Y|X_i=x_i}$ and $x^n \in \mathcal{T}_\varepsilon^{(n)}$

$$\mathbb{P}\left[Y^n \in \mathcal{T}_{\varepsilon'}^{(n)}(Y|x^n)\right] \longrightarrow 1 \quad \text{as } n\to\infty \quad \forall \varepsilon' > \varepsilon$$

This is the main reason why we switched to $\mathcal{T}_\varepsilon^{(n)}$



Type class: $T_P^{(n)} = \{ x^n : \pi(x|x^n) = P(x) \}$

Both $\mathcal{T}_\varepsilon^{(n)}$ and $A_\varepsilon^{(n)}$ are unions of type classes.

Proof of 7.a.

$$\frac{\pi(x,y|x^n,y^n)}{\pi(x|x^n)} = \frac{1}{|\{i : x = x_i\}|} \sum_{i:X_i=x} \underbrace{\mathbb{1}\{(x_i, Y_i) = (x,y)\}}_{1(Y_i=y)}$$

$$\longrightarrow \mathbb{E}_{P_{Y|X=x}} 1(Y_i = y) \qquad \text{LLN because } |\{i : x_i = x\}|$$

$$= P_{Y|X}(y|x)$$

$$\Rightarrow \mathbb{P}\left[\frac{\pi(x,y|x^n,y^n)}{\pi(x|x^n)} \geq P_{Y|X}(y|x)(1+\bar{\varepsilon})\right] \longrightarrow 0 \qquad \forall \bar{\varepsilon} > 0$$

$$\Rightarrow \mathbb{P}\left[\pi(x,y|x^n,y^n) \geq P_{Y|X}(y|x)\, P_X(x)\,(1+\bar{\varepsilon})\left(1+\frac{\varepsilon}{H(X)}\right)\right] \longrightarrow 0$$

Let $\bar{\varepsilon}$ be small enough

s.t. $(1+\bar{\varepsilon})\left(1+\frac{\varepsilon}{H(X)}\right) < 1 + \frac{\varepsilon'}{H(X)}$

7.b. $\left|T_{\varepsilon'}^{(n)}(Y(x^n))\right| \in \left[(1-\varepsilon')\, 2^{-n(H(Y|X)+2\varepsilon')},\ 2^{-n(H(Y|X)-2\varepsilon')}\right]$

n large enough

proof note: If $(x^n, y^n) \in T_{\varepsilon'}^{(n)}(Y(x^n))$; $P_{Y^n|X^n}(y^n|x^n) \in \left[2^{-n(H(Y|X)+2\varepsilon')},\ 2^{-n(H(Y|X)-2\varepsilon')}\right]$

$$P_{Y^n|X^n}(y^n|x^n) = \frac{P_{X^nY^n}(x^n,y^n)}{P_{X^n}(x^n)} \leq \frac{2^{-n(H(X;Y)-\varepsilon')}}{2^{-n(H(X)+\varepsilon')}}$$

7.c.

If $x^n \in T^{(n)}$ and Y^n i.i.d. $\sim P_Y$

$$\mathbb{P}\left[Y^n \in T_{\varepsilon'}^{(n)}(Y|x^n)\right] \in \left[(1-\varepsilon')\, 2^{-n(I(X,Y)+4\varepsilon')},\ 2^{-n(I(X;Y)-4\varepsilon')}\right]$$

n large enough

holds for $A_{\varepsilon'}^{(n)}$ as well.

7.c* Given P_{XYZ}, If $(x^n, y^n) \in T_{\varepsilon}^{(n)}$ and $Z^n \sim \prod_{i=1}^{n} P_{Z_i|Y_i=y_i}$

$$\mathbb{P}\left[Z^n \in T_{\varepsilon'}^{(n)}(Z|x^n,y^n)\right] \in \left[(1-\varepsilon')\, 2^{-n(I(X;Z|Y)+4\varepsilon')},\ 2^{-n(I(X;Z|Y)-4\varepsilon')}\right]$$

n large enough

Note: If $X - Y - Z$

$$\mathbb{P}\left[Z^n \in T_{\varepsilon'}^{(n)}(Z|x^n,y^n)\right] \longrightarrow 1$$

R-D theorem (converse)

Suppose $\exists n, f, g$ at rate R s.t. $\mathbb{E}[d(X^n, \hat{X}^n)] \leq D \Rightarrow X^n - M - \hat{X}^n \quad M \in [2^{nR}]$

$nR \geq H(M)$

$\geq I(X^n; \hat{X}^n)$ D.P.I.

$= \sum_{i=1}^{n} I(X_i, \hat{X}^n | X^{i-1})$

another way

$\sum_{i=1}^{n} I(X_i; \hat{X}^n X^{i-1})$

$\geq \sum_{i=1}^{n} I(X_i; \hat{X}^n)$ because $X_i \perp\!\!\!\perp X^{i-1}$

$\geq \sum_{i=1}^{n} I(X_i, \hat{X}_i)$

Let $T \sim \text{unif}\{1,\dots,n\}$

$T \perp\!\!\!\perp (X^n, M, \hat{X}^n)$

$$\sum_{i=1}^{n} I(X_i;\hat{X}_i)$$
$$= nI(X_T;\hat{X}_T|T)$$
$$\geq nI(X_T;\hat{X}_T)$$

$X_T \perp\!\!\!\perp T$ as X_i are i.i.d.

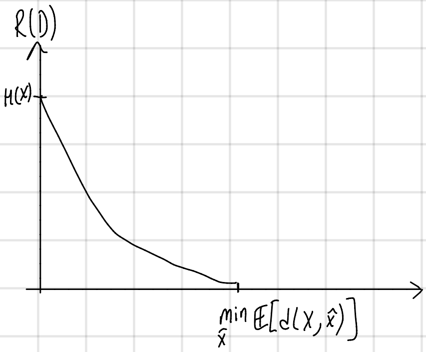
Lastly,

define $X = X_T$, $\hat{X} = \hat{X}_T$, notice $X \sim P_X$

Also " $\mathbb{E}[d(X,\hat{X})] = \mathbb{E}[\mathbb{E}_{T(X,\hat{X})|X^n\hat{X}^n}[d(X,\hat{X})]]$

$$= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} d(X^n,\hat{X}^n)\right]$$
$$\leq D$$

10/13/2016
Thursday



Let (R, D) be in the <u>interior</u>:

$\Longrightarrow \exists P_{\hat{X}|X}$ s.t.

$I(X;\hat{X}) < R$
$\mathbb{E}[d(X,\hat{X})] < D$

Codebook: Derive $P_{\hat{X}}(\hat{x}) = \sum_x P_X(x) P_{\hat{X}|X}(\hat{x}|x)$

$C = \{\hat{X}^n(m)\}_{m=1}^{2^{nR}}$ where $\hat{X}^n(m)$ are i.i.d. $\sim P_{\hat{X}}$ $\forall m$ and independent.

Encoder: Choose $\varepsilon > 0$, find any m s.t. $(X^n, \hat{X}(m)) \in \mathcal{T}_\varepsilon^{(n)}$ Transmit m (error occurs only if $\nexists m$!)
X^n ↳ observed.
If error, chose any m.

↳ complexity is in encoder (cf. Channel coding thm proof, where complexity is in decoder)

Decoder: Reconstruct $\hat{X}(m)$